PROBLEM OF THE WEEK #5
(Fall 2016)

A certain online prediction market offers contracts whose values depend on future events. For example, you can buy shares of a contract called MERKEL2017 in the market. If Angela Merkel is reëlected Chancellor of Germany in 2017, you will receive \$1 for each share of MERKEL2017 you own – otherwise, these shares are worthless. After the election, the shares disappear.

The site will charge you a brokerage fee of 10% of any profit you make on your shares. [This is 10% of the profit, not the gross payout. No brokerage fee applies if you lose money on the shares.]

Additionally, if you cash out, the site charges a processing fee of 5% of your withdrawal.

Let p denote Dr. Black's best estimate of the probability that Chancellor Merkel will be reëlected in 2017. What is the most that Dr. Black might be willing to pay for a share of MERKEL2017?

Solution:

If Dr. Black is willing to pay x dollars for a share of MERKEL2017, then $x \leq \dfrac{171p}{200 - 19p}$.

Proof. Suppose Dr. Black puts x into a new account at the site, and then pays x for one share of MERKEL2017. [All prices are given in dollars.]

If Merkel is reëlected, Dr. Black will be paid 1 for his share of MERKEL2017. His profit on this share is $1 - x$, and he is immediately charged a brokerage fee of $0.1(1 - x)$. His account balance is therefore $1 - 0.1(1 - x) = 0.9 + 0.1x$. When he collects this money from the site, he pays a 5% processing fee, so he receives $0.95(0.9 + 0.1x) = 0.855 + 0.095x$.

On the other hand, if Merkel is not reëlected, Dr. Black receives nothing. He therefore expects to receive $p(0.855 + 0.095x)$. If he is willing to pay x for this return, then

$$
\begin{aligned}
x &\leq p(0.855 + 0.095x) \\
x(1 - 0.095p) &\leq 0.855p \\
x &\leq \frac{0.855p}{1 - 0.095p} \\
x &\leq \frac{171p}{200 - 19p}
\end{aligned}
$$

$\square$

Remark. Dr. Black should not really pay $\dfrac{171p}{200 - 19p}$ for a share of MERKEL2017: he should demand a discount, for reasons like risk premium and opportunity cost. This question can be analyzed more deeply!

Source: "How it Works." `https://www.predictit.org/About/HowItWorks`