PROBLEM OF THE WEEK #4

(Fall 2016)

Let $x_1, x_2, \ldots, x_n$ be equally spaced points on the unit circle. What do you get if you multiply together the distances from x_1 to the other $n-1$ points?

Solution:

The product equals n.

Proof. Working in the complex plane, we can rotate our points about the origin without changing the chord lengths. Thus our product is equal to

$$\prod_{k=1}^{n-1} \left| 1 - e^{2\pi ki/n} \right|$$

$$= \lim_{z \to 1} \prod_{k=1}^{n-1} \left| z - e^{2\pi ki/n} \right|$$

$$= \lim_{z \to 1} \left[\prod_{k=0}^{n-1} \left| z - e^{2\pi ki/n} \right| \right] / |z - 1|$$

$$= \lim_{z \to 1} \left| \frac{z^n - 1}{z - 1} \right|$$

$$= \lim_{z \to 1} \sum_{d=0}^{n-1} \left| z^d \right|$$

$$= \sum_{d=0}^{n-1} 1$$

$$= n.$$

$\square$

Source: Mudd Math Fun Facts. "Chords of a Unit Circle." `https://www.math.hmc.edu/funfacts/ffiles/20001.1-2-3.shtml`