UNIVERSITY OF WISCONSIN
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #3

(Fall 2016)

Five spheres are stacked up inside a cone; each sphere is tangent to the cone and to the next sphere(s) in the stack. The smallest sphere has radius 144, and the largest has radius 196. What is the radius of the middle sphere?

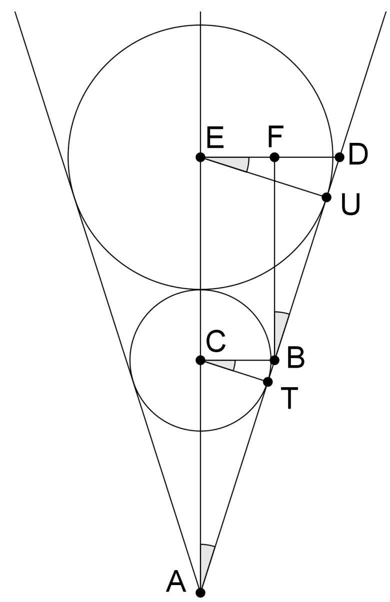
Solution:
The radius of the middle sphere is 168.

Proof. Consider the relationship of a pair of spheres, tangent to each other and to the cone. We name the vertex of the cone A, the center of the smaller sphere C, and the center of the larger sphere E. We study a cross-section through A, C, and E (see figure).
In this cross-section, the smaller sphere is tangent to the cone at T; the larger sphere is tangent to the same side of the cone at U. The perpendiculars to CE through C and E meet that side of the cone at B and D respectively. The parallel to CE through B intersects DE at F.
Let θ denote the common measure of $\angle BAC$, $\angle BCT$, $\angle DEU$, and $\angle DBF$. Let r_1 and r_2 denote the radii of the spheres, with $r_1 < r_2$. Radii of circles are perpendicular to tangent lines, so, by right-triangle trigonometry, we know that $BC = r_1 \sec\theta$ and $DE = r_2 \sec\theta$. From the right triangle $\triangle BDF$, we have

$$\tan\theta = \frac{DF}{BF} = \frac{DE - BC}{CE} = \frac{(r_2 - r_1)\sec\theta}{r_1 + r_2}.$$

Therefore:

$$\begin{aligned}
(r_1 + r_2)\tan\theta &= (r_2 - r_1)\sec\theta \\
(r_1 + r_2)\sin\theta &= (r_2 - r_1) \\
r_1(1 + \sin\theta) &= r_2(1 - \sin\theta) \\
\frac{r_2}{r_1} &= \frac{1 + \sin\theta}{1 - \cos\theta}
\end{aligned}$$



Thus the ratio k between the radii of adjacent circles depends only on the base angle of the cone.
Hence $196 = 144k^4$, so $k^4 = \frac{196}{144} = \frac{49}{36}$, and the middle sphere has radius $144k^2 = 144 \cdot \frac{7}{6} = 168$.
[This is the *harmonic mean* of 144 and 196.] □

Source: "Nick's Mathematical Puzzles #37: Five Marbles." Online at `http://www.qbyte.org/puzzles/puzzle04.html`.