University of Wisconsin
PLATTEVILLE
DEPARTMENT OF MATHEMATICS

Problem of the Week #2

(Fall 2016)

Find integers x and y such that

$$\begin{cases} x^2 - 16x + 3y = 20, \\ y^2 + 4y - x = -12. \end{cases}$$

Solution:
Complete the square:

$$\begin{cases} (x-8)^2 + 3y = 84, \\ (y+2)^2 - x = -8. \end{cases}$$

Let $a = x - 8$ and $b = y + 2$, and simplify:

$$\begin{cases} a^2 + 3b = 90, \\ b^2 - a = 0. \end{cases}$$

Thus $a = b^2$, and therefore:

$$\begin{aligned} b^4 + 3b &= 90 \\ b^4 + 3b - 90 &= 0 \\ (b-3)(b^3 + 3b^2 + 9b + 30) &= 0 \end{aligned}$$

So we get a solution by setting $b = 3$, which yields $a = b^2 = 9$ and $\boxed{(x, y) = (17, 1)}$.

In fact, this is the only solution. We know this because y is an integer only if b is an integer, and $f(b) = b^3 + 3b^2 + 9b + 30$ has no integer roots. To see this, observe that

$$f'(b) = 3b^2 + 6b + 9 = 3(b+1)^2 + 6 > 0,$$

so f is increasing. On the other hand, $f(-4) = -22$, and $f(-3) = 3$. By Rolle's theorem and the intermediate value theorem, it follows that f has a unique real root, which lies in the interval $(-4, -3)$, and is therefore not an integer.

Source: Andreescu, Tito, and Jonathan Kane. *Purple Comet! Math Meet: The first ten years.* XYZ Press, 2013 (pp. 15, 128).