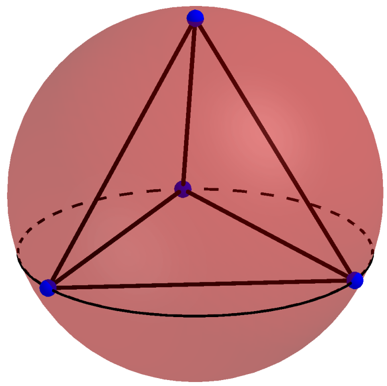
Problem of the Week #1

(Fall 2016)

How long are the edges of a regular tetrahedron inscribed in the unit sphere?



Solution:
The edge length is $\frac{2\sqrt{6}}{3}$.

Proof. Set up a coordinate system so that one vertex of the tetrahedron is at $(0,0,1)$, and another is at $(\sin\phi, 0, \cos\phi)$. Then the other two vertices will be at $(-\frac{1}{2}\sin\phi, \pm\frac{\sqrt{3}}{2}\sin\phi, \cos\phi)$. [We can get these points using spherical coordinates, with $\theta \in \{0, 2\pi/3, 4\pi/3\}$.]
If we call the edge length D, then, by the distance formula:

$$\begin{aligned} D^2 &= \frac{9}{4}\sin^2\phi + \frac{3}{4}\sin^2\phi = 3\sin^2\phi, \text{ and} \\ D^2 &= \sin^2\phi + (1-\cos\phi)^2 = 2 - 2\cos\phi. \end{aligned}$$

Therefore:

$$\begin{aligned} 3(1-\cos^2\phi) &= 2 - 2\cos\phi \\ 3\cos^2\phi - 2\cos\phi - 1 &= 0 \\ (3\cos\phi + 1)(\cos\phi - 1) &= 0 \end{aligned}$$

Since $\phi \in (0,\pi)$, we have $\cos\phi = -1/3$. This means $D^2 = 8/3$, and $D = \sqrt{\frac{8}{3}} = \frac{2\sqrt{6}}{3}$. $\square$